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1989 J. Phys. A: Math. Gen. 22 5297

(http://iopscience.iop.org/0305-4470/22/24/018)

J. Phys. A: Math. Gen. **22** (1989) 5297-5301. Printed in the UK

COMMENT

On dimensional regularisation and critical properties of systems with long-range correlated impurities

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Received 4 July 1989

Abstract. We generalise the method of renormalisation by minimal subtraction of dimensional poles to the case of having interacting terms in the Hamiltonian with different critical dimensionalities d_c and $d'_c = d_c - \sigma$, where σ is a variable parameter of the theory. Results to one-loop order for the random m-component ferromagnet with long-range correlated impurities agree with the theory of Weinrib and Halperin.

We comment in this paper on the use of the method of renormalised perturbation theory with dimensional regularisation and minimal subtraction of dimensional poles (Amit 1978) to study critical properties of systems with long-range interactions. Although we limit the discussion here to a random m-component ferromagnet with long-range correlated impurities, as described by the Hamiltonian of Weinrib and Halperin (1983), our results can be applied to a wider class of systems having two interacting terms with different critical dimensionalities, d_c and $d'_c = d_c - \sigma$ where σ is a variable parameter of the theory.

Systems described by a Hamiltonian with long-range interactions in their quadratic or free term have been extensively studied when the interacting term is that of a φ^4 theory (Sak 1977, Gusmão and Theumann 1983), a φ^3 theory (Theumann and Gusmão 1985) or for the spin-glass theory (Chang and Sak 1984).

In the case of having only one interacting term, the critical dimensionality is the number of space dimensions at which the theory is renormalisable, which means that the degree of divergence of the integrals entering the calculation of a given vertex function is independent of the order in perturbation theory. A systematic approach to the problem is then to expand around the critical dimensionality and to take $\varepsilon = d_c - d$ as the expansion parameter. To renormalise the theory by using the method of dimensional regularisation and minimal subtraction, the integrals are performed at a dimensionality $d < d_c$ when they are convergent, and their singular behaviour for $d = d_c$ is reflected by the presence of poles in ε that should be 'minimally subtracted' order by order in perturbation theory (Amit 1978). The question then poses itself as to which is the correct expansion parameter in the case of having two interacting terms with different critical dimensionalities. It was shown by Weinrib and Halperin that the renormalisation group recursion relations lead to a double expansion in $\varepsilon = d_c - d$ *and* $\varepsilon' = d'_c - d$.

† Supported by the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and Financiadora de Estudos e Projectos (FINEP).

The purpose of the present comment is to show how the same result can be obtained within the method of renormalisation with minimal subtraction of dimensional poles. We propose that the generalised procedure should be to subtract the dimensional poles in ε *and* in ε', while terms of the type $(\varepsilon/\varepsilon')$ must be considered 'regular' and do not need to be subtracted.

We apply these ideas to the m-vector model with a random temperature term (Weinrib and Halperin 1983) $\delta r(\boldsymbol{x})$, with zero mean and Gaussian correlations:

$$\langle \delta r(\boldsymbol{x})\delta r(\boldsymbol{x}')\rangle = g_s\delta(\boldsymbol{x}-\boldsymbol{x}') + g_1|\boldsymbol{x}-\boldsymbol{x}'|^{-d-\sigma} \tag{1}$$

that through the use of the replica trick can be cast into the effective Hamiltonian:

$$H_{\text{eff}} = \frac{1}{2}\int \mathrm{d}\boldsymbol{k}(r_0+k^2)\sum_{\alpha}\boldsymbol{\phi}_\alpha(\boldsymbol{k})\cdot\boldsymbol{\phi}_\alpha(-\boldsymbol{k}) + \sum_{\alpha,\beta}\frac{1}{4!}\int \mathrm{d}\boldsymbol{k}_1\,\mathrm{d}\boldsymbol{k}_2 d\boldsymbol{q}(g_0\delta_{\alpha\beta}-g_s-g_1q^\sigma)$$
$$\times[\boldsymbol{\phi}_\alpha(-\boldsymbol{k}_1+\boldsymbol{q})\cdot\boldsymbol{\phi}_\alpha(\boldsymbol{k}_1)][\boldsymbol{\phi}_\beta(-\boldsymbol{k}_2+\boldsymbol{q})]\cdot\boldsymbol{\phi}_\beta(\boldsymbol{k}_2)] \tag{2}$$

where

$$\boldsymbol{\varphi}_\alpha(\boldsymbol{k})\cdot\boldsymbol{\varphi}_\alpha(\boldsymbol{k}') = \sum_{i=1}^{m}\varphi_{\alpha i}(\boldsymbol{k})\cdot\varphi_{\alpha i}(\boldsymbol{k}') \tag{3}$$

and α is the replica index running up to n, in the limit $n\to 0$.

We now follow closely Amit (1978) in the analysis of the primitive divergencies that occur in perturbation theory. Elementary power counting tells us that the interactions in (2) (see figure 1) can be written as $g_i = u_i\kappa^{4-d}$, $i=0$ or s, and $g_1 = u_1\kappa^{4-d-\sigma}$ where κ is the scale parameter with dimensions of an inverse length and the u are dimensionless couplings.

The primitive divergence of a diagram in the perturbation expansion of a vertex function is calculated as usual, only that now it should be counted that a long-range interaction *integrated* in a loop adds a factor Λ^σ from (2), where Λ is an ultraviolet cut-off. Hence we obtain that a graph of nth order for a vertex function $\Gamma^{(E)}(k_i)$, with E external legs of external momentum $\boldsymbol{k}_i$ and μ *integrated* long-range interactions will diverge asymptotically as Λ^δ, where

$$\delta = -n(4-d)+\mu\sigma+d-E(d/2-1). \tag{4}$$

For instance, the diagrams in figure 3(d) and in figure 3(e) below will have $\mu=2$ when both interactions correspond to u_1, while $\mu=1$ if one interaction is of long-range

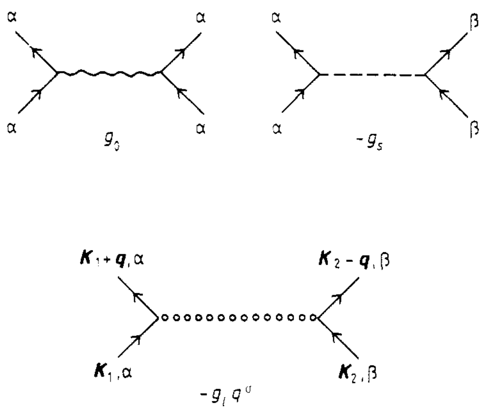

Figure 1. The basic quartic interactions.

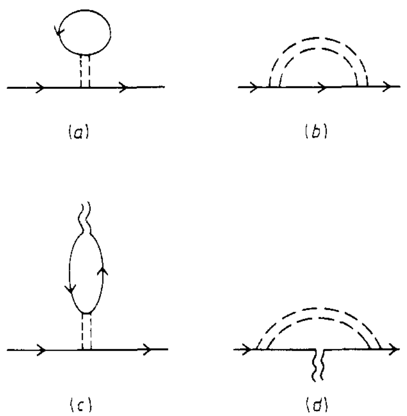

Figure 2. (a), (b) one-loop order contribution to $\Gamma^{(2)}_{\alpha\alpha}(\boldsymbol{k})$; ($c$), ($d$) one-loop order contributions to $\Gamma^{(2,1)}_{\alpha\alpha}(\boldsymbol{k}, \boldsymbol{q})$. Here the double pointed line indicates any of the three interactions of figure 1 while the double wavy line indicates an insertion. Diagrams (a) and (c) differ from zero only for u_0, in the limit $n = 0$.

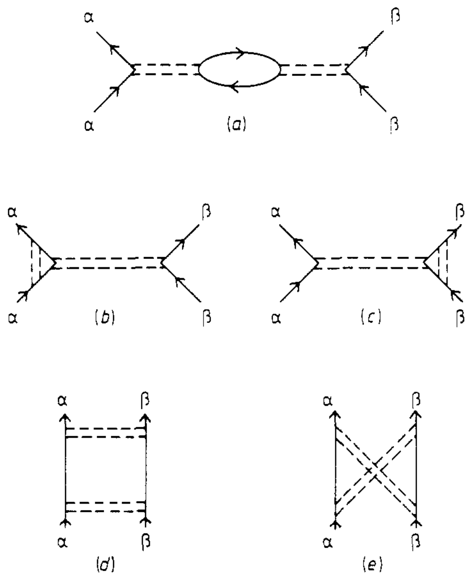

Figure 3. One-loop corrections to $\Gamma^{(4)}_{\alpha\beta}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{q})$. The double pointed lines indicate any of the three interactions in figure 1. Diagram(a) differs from zero only for u_0 in the limit $n = 0$.

type while the other corresponds to u_s or u_0, and $\mu = 0$ if neither of them is long range. The diagrams in figure 3(b) and in figure 3(c) can have at most $\mu = 1$. We note also that usually non-integrated interactions are linked to closed loops as in figure 2(a), figure 2(c) and figure 3(a), and these diagrams will vanish for u_s or u_l in the limit $n = 0$. We may mention here that if we had a long-range interaction term $g_0' \delta_{\alpha\beta} q^{\sigma'}$ present in the Hamiltonian of (2), this class of diagram would not vanish and would bring problems for zero momentum transfer when $\sigma' < 0$.

It is clear from (4) that the only way to achieve an expansion where the primitive divergence of a vertex function is independent of the order in perturbation theory is by using $\varepsilon = 4 - d$ *and* σ as expansion parameters. By performing the integrals at $d < 4$ and $\sigma < 0$, when they are convergent the ultraviolet singularities will appear as dimensional poles in ε and σ, or in $\varepsilon' = \varepsilon - \sigma$.

We propose here that a correct renormalisation of the theory can be achieved by minimally subtracting the dimensional poles in ε *and* σ (or ε') by means of an expansion in powers of dimensionless potentials λ_0, λ_s, and λ_1. Also we propose that terms of the type $(\varepsilon/\varepsilon')^s$ should be considered as regular, for s a positive or negative integer, and as such they should not be subtracted.

As an example we show below that the renormalisation of the theory with the Hamiltonian of (2) to one-loop order reproduces the results of Weinrib and Halperin (1983) in a double expansion in ε and σ. The diagrams shown in figure 2 and figure 3 give for the two-point vertex $\Gamma^{(2)}_{\alpha\alpha}$, for the two-point function with a φ^2-insertion $\Gamma^{(2,1)}_{\alpha\alpha}$ and for the four-point vertex $\Gamma^{(4)}_{\alpha\beta}$, at the critical $r_0 = 0$:

$$\Gamma^{(2)}_{\alpha\alpha}(\boldsymbol{k}) = k^2 - \frac{\kappa^2}{6}[(m+2)u_0 L_0 - 2u_s L_0 - 2u_l L_1] \tag{5}$$

$$\Gamma^{(2,1)}_{\alpha\alpha}(\boldsymbol{k}, \boldsymbol{q}) = 1 - \tfrac{1}{6}[(m+2)u_0 I_0 - 2u_s I_0 - 2u_l I_1] \tag{6}$$

$$\begin{aligned}\Gamma^{(4)}_{\alpha\beta}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{q}) &= \delta_{\alpha\beta}\kappa^{\varepsilon}\{u_0 - \tfrac{1}{6}[u_0^2(m+8)I_0 - u_0 u_s 12 I_0 - 12 u_0 u_l I_1]\} \\ &\quad - \kappa^{\varepsilon}\{u_s - \tfrac{1}{3}[u_0 u_s(m+2)I_0 - 4u_s^2 I_0 - 6u_s u_l I_1 - u_l^2 2 I_2]\} \\ &\quad - \kappa^{\varepsilon-\sigma}q^{\sigma}\{u_l - \tfrac{1}{3}[u_0 u_l(m+2)I_0 - 2u_l u_s I_0 - 2u_l^2 I_1]\}\end{aligned} \tag{7}$$

where I_μ (L_μ) are dimensionless integrals with two (one) propagators and with μ long-range vertices, $\mu = 0$, 1 or 2. A factor S_d, the surface of the unit sphere, is absorbed in the definition of the interactions.

The calculation of L_0 is standard and its singular part vanishes at the critical value $r_0 = 0$ (Amit 1978), but we have to calculate:

$$L_1(\boldsymbol{k}) = \frac{1}{S_d}\int \frac{\mathrm{d}\boldsymbol{p}}{(2\pi)^d}\frac{p^{\sigma}}{(\boldsymbol{p}+\boldsymbol{k})^2} = k^{d-2+\sigma}\frac{\Gamma(1-\sigma/2)}{\Gamma(-\sigma/2)}B\left(\frac{d}{2}-1, \frac{d+\sigma}{2}\right)B\left(\frac{d}{2}, 1-\frac{d+\sigma}{2}\right). \tag{8}$$

From (8) we find the singular behaviour when $d = 4 - \varepsilon$:

$$[L_1(\boldsymbol{k})]_{\text{sin}} = \frac{k^2}{2}\frac{\sigma}{\varepsilon - \sigma}. \tag{9}$$

Then apparently we would obtain a singular contribution for the k^2 term already at one-loop order, but this singularity appears multiplied by σ, so the contribution of (9) is $\mathrm{O}[\sigma/(\varepsilon-\sigma)]$ and it should be considered as 'regular'. The calculation of the singular part of I_μ is standard and we obtain:

$$[I_\mu]_{\text{sin}} = \frac{2}{\varepsilon - \mu\sigma}. \tag{10}$$

To remove the dimensional poles of a relevant vertex functions in (5)-(7), we renormalise as

$$Z_\varphi \Gamma^{(2)}_{\alpha\alpha} = \Gamma^{(2)}_{R\alpha\alpha} \qquad Z_{\varphi^2}\Gamma^{(2,1)}_{\alpha\alpha} = \Gamma^{(2,1)}_{R\alpha\alpha} \qquad Z_\varphi^2 \Gamma^{(4)}_{\alpha\beta} = \Gamma^{(4)}_{R\alpha\beta} \tag{11}$$

by means of dimensionless functions $u_i(\{\lambda_j\}, \varepsilon, \sigma)$, $Z_\varphi(\{\lambda_j\}, \varepsilon, \sigma)$, $Z_{\varphi^2}(\{\lambda_j\}, \varepsilon, \sigma)$, of the renormalised couplings $\{\lambda_j\} i, j = 0$, s or 1. From the discussion following (9) we obtain immediately $Z_\varphi = 1$ while:

$$Z_{\varphi^2} = 1 + \frac{1}{3}\left((m+2)\frac{\lambda_0}{\varepsilon} - 2\frac{\lambda_s}{\varepsilon} - 2\frac{\lambda_1}{\varepsilon - \sigma}\right) \tag{12}$$

$$u_0 = \lambda_0 + \tfrac{1}{3}(m+8)\frac{\lambda_0^2}{\varepsilon} - \frac{4}{\varepsilon}\lambda_0\lambda_s - \frac{4}{\varepsilon - \sigma}\lambda_0\lambda_1$$

$$u_s = \lambda_s + \tfrac{2}{3}(m+2)\frac{1}{\varepsilon}\lambda_0\lambda_s - \tfrac{8}{3}\lambda_s^2\frac{1}{\varepsilon} - \frac{4}{\varepsilon - \sigma}\lambda_s\lambda_l - \frac{4}{3}\frac{1}{\varepsilon - 2\sigma}\lambda_1^2 \tag{13}$$

$$u_1 = \lambda_1 + \tfrac{2}{3}(m+2)\frac{1}{\varepsilon}\lambda_0\lambda_1 - \tfrac{4}{3}\lambda_1\lambda_s\frac{1}{\varepsilon} - \frac{4}{3}\frac{1}{\varepsilon - \sigma}\lambda_1^2.$$

From (13) we obtain, by varying λ_i at constant dimensional couplings $\{g_j\}$, the β functions $\beta_i = [\kappa\partial\lambda_i/\partial\kappa]_{g_i}$:

$$\begin{aligned}
\beta_0 &= -\varepsilon\lambda_0 + \tfrac{1}{3}(m+8)\lambda_0^2 - 4\lambda_0\lambda_1 - 4\lambda_0\lambda_s \\
\beta_s &= -\varepsilon\lambda_s + \tfrac{2}{3}(m+2)\lambda_0\lambda_s - \tfrac{8}{3}\lambda_s^2 - 4\lambda_s\lambda_l - \tfrac{4}{3}\lambda_1^2 \\
\beta_1 &= -(\varepsilon - \sigma)\lambda_1 + \tfrac{2}{3}(m+2)\lambda_0\lambda_1 - \tfrac{4}{3}\lambda_s\lambda_1 - \tfrac{4}{3}\lambda_1^2.
\end{aligned} \tag{14}$$

The three β functions in (14) coincide with the differential recursion relations of Weinrib and Halperin (1983) if we redefine the interactions by a factor and take $\kappa = l^{-1}$. To one-loop order we also obtain for the exponents:

$$\eta = \kappa\frac{\partial}{\partial\kappa}[\ln Z_\varphi] = 0$$

$$\begin{aligned}
\nu^{-1} - 2 &= \kappa\frac{\partial}{\partial\kappa}[\ln Z_{\varphi^2}] \\
&= -\tfrac{1}{3}(m+2)\lambda_0^* + \tfrac{2}{3}\lambda_s^* + \tfrac{2}{3}\lambda_1^*
\end{aligned} \tag{15}$$

in agreement with the results of Weinrib and Halperin.

References

Amit D J 1978 *Field Theory, the Renormalization Group and Critical Phenomena* (New York: McGraw-Hill)
Chang M and Sak J 1984 *Phys. Rev.* B **29** 2652-4
Gusmão M A and Theumann W K 1983 *Phys. Rev.* B **28** 6545-7
Sak J 1977 *Phys. Rev.* B **15** 4344
Theumann W K and Gusmão M A 1985 *Phys. Rev.* B **31** 379-88
Weinrib A and Halperin B I 1983 *Phys. Rev.* B **27** 413-27